Modeling Dividends and Other Distributions
Retaining And Reinvesting Dividends

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Company capital that is over and above what is needed to finance the company's current operations is paid out over time as dividends and/or other distributions. This excess capital can be distributed to shareholders, retained by the company, or both. In this white paper we model the cash account used to accumulate and reinvest dividends and other distributions retained by the company. To that end we will use the following hypothetical problem...

Our Hypothetical Problem

We are tasked with building a model to model the dividends and other distributions cash account. We are given the following go-forward model assumptions...

Table 1: Go-Forward Model Assumptions

Description	Value
Asset value at time zero ($)	1,000,000
Expected return - mean (%)	10.00
Expected return - volatility (%)	20.00
Dividends and other distributions (%)	4.00
Risk-free interest rate (%)	3.00

Our task is to answer the following questions given that the random draw from a normal distribution with mean zero and variance one was 0.75...

Question 1: What is asset value at the end of year 5?

Question 2: What are cumulative dividends and distributions over the time interval $[0, 5]$?

Question 3: What is the dividends and distributions cash account balance at the end of year 5?

Modeling Asset Value Over Time

We will define the variable A_t to be asset value at time t, the variable μ to be the expected rate of return, the variable ϕ to be the dividend yield, the variable σ to be expected volatility, and the variable δW_t to be the change in the underlying brownian motion over time. The stochastic differential equation for the change in asset value over the time interval $[t, t + \delta t]$ is...

$$\delta A_t = \mu\, A_t\, \delta t - \phi\, A_t\, \delta t + \sigma\, A_t\, \delta W_t \;\; \text{...where...} \;\; \delta W_t \sim N\Big[0, \delta t\Big] \tag{1}$$

Note that we can normalize and rewrite Equation (1) above as...

$$\delta A_t = \mu\, A_t\, \delta t - \phi\, A_t\, \delta t + \sigma\, A_t \sqrt{t}\, Z \;\; \text{...where...} \;\; Z \sim N\Big[0, 1\Big] \tag{2}$$

The solution to Equation (2) above is the equation for random asset value at the end of time t, which is...

$$A_t = A_0 \,\text{Exp}\left\{\left(\mu - \phi - \frac{1}{2}\sigma^2\right) t + \sigma \sqrt{t}\, Z\right\} \;\; \text{...where...} \;\; Z \sim N\Big[0, 1\Big] \tag{3}$$

We will define the variable θ to be the normally-distributed random annualized rate of return. Using Equation (3) above the equation for the variable θ at time t is...

$$\theta = \left[\left(\mu - \phi - \frac{1}{2}\,\sigma^2\right)t + \sigma\,\sqrt{t}\,Z\right]\Big/\,t \;\;\text{...where...}\;\; Z \sim N\Big[0,1\Big] \tag{4}$$

Using Equation (4) above we can rewrite random asset value Equation (3) above as...

$$A_t = A_0\,\text{Exp}\Big\{\theta\,t\Big\} \;\;\text{...where...}\;\; \theta \sim N\left[\left(\mu - \phi - \frac{1}{2}\,\sigma^2\right)t,\;\sigma^2 t\right] \tag{5}$$

Note that random asset value in Equation (5) above is a lognormally-distributed random variable and as such has the following expectation...

$$\mathbb{E}\Big[A_t\Big] = A_0\,\text{Exp}\left\{\text{Mean} + \frac{1}{2}\,\text{Variance}\right\} = A_0\,\text{Exp}\left\{\left(\mu - \phi - \frac{1}{2}\,\sigma^2\right)t + \frac{1}{2}\,\sigma^2 t\right\} = A_0\,\text{Exp}\Big\{(\mu - \phi)\,t\Big\} \tag{6}$$

Cumulative Dividends and Distributions

We will define the variable D_t to be random cumulative dividends and distributions over the time interval $[0,t]$. Using Equation (5) above the equation for random cumulative dividends and distributions is...

$$D_t = \int_0^t \phi\,A_u\,\delta u = \phi\int_0^t A_0\,\text{Exp}\Big\{\theta\,u\Big\}\,\delta u \;\;\text{...where...}\;\; \theta \sim N\left[\left(\mu - \phi - \frac{1}{2}\,\sigma^2\right)t,\;\sigma^2 t\right] \tag{7}$$

The solution to the integral in Equation (7) above is...

$$\int_0^t A_0\,\text{Exp}\Big\{\theta\,u\Big\}\,\delta u = \frac{1}{\theta}\,A_0\left[\text{Exp}\Big\{\theta\,t\Big\} - 1\right] \tag{8}$$

Using Equations (7) and (8) above the equation for random cumulative dividends and distributions over the time interval $[0,t]$ is...

$$D_t = \int_0^t \phi\,A_u\,\delta u = \frac{\phi}{\theta}\,A_0\left[\text{Exp}\Big\{\theta\,t\Big\} - 1\right] \tag{9}$$

Retaining and Reinvesting Dividends and Distributions

We will define the variable C_t to be the random cash account balance at time t and the variable α to be the risk-free rate of interest. The cash account is used to deposit dividends and distributions retained by the company (rather than paying them out to shareholders). The cash account is assumed to earn the risk-free rate of interest. Using Equation (9) above the equation for the random cash account balance is...

$$C_t = \int_0^t \phi\,A_u\,\text{Exp}\Big\{\alpha\,(t-u)\Big\}\,\delta u = \phi\,\text{Exp}\Big\{\alpha\,t\Big\}\int_0^t A_u\,\text{Exp}\Big\{-\alpha\,u\Big\}\,\delta u \tag{10}$$

Using Equation (5) above we can rewrite Equation (10) above as...

$$C_t = \phi\,\text{Exp}\Big\{\alpha\,t\Big\}\int_0^t A_0\,\text{Exp}\Big\{\theta\,u\Big\}\,\text{Exp}\Big\{-\alpha\,u\Big\}\,\delta u = \phi\,\text{Exp}\Big\{\alpha\,t\Big\}\int_0^t A_0\,\text{Exp}\Big\{(\theta - \alpha)\,u\Big\}\,\delta u \tag{11}$$

The solution to the integral in Equation (11) above is...

$$\int_0^t A_0\,\text{Exp}\Big\{(\theta - \alpha)\,u\Big\}\,\delta u = \frac{1}{\theta - \alpha}\,A_0\left[\text{Exp}\Big\{(\theta - \alpha)\,t\Big\} - 1\right] \tag{12}$$

Using Equation (12) above we can rewrite random cash account Equation (11) above as...

$$C_t = \frac{\phi}{\theta - \alpha}\,A_0\,\text{Exp}\Big\{\alpha\,t\Big\}\left(\text{Exp}\Big\{(\theta - \alpha)\,t\Big\} - 1\right) \tag{13}$$

The Answers To Our Hypothetical Problem

Using Table 1 above our continuous-time model parameters are...

Table 2: Model Parameters

Symbol	Description	Value	Calculation
A_0	Asset value at time zero	1,000,000	
μ	Expected return - mean	0.0953	$\ln(1+0.10)$
σ	Expected return - volatility	0.2000	
ϕ	Dividends and other distributions	0.0392	$\ln(1+0.04)$
α	Risk-free interest rate	0.0296	$\ln(1+0.03)$
t	Term in years	5.0000	
Z	Normally-distributed random draw	0.7500	

Using Equation (4) above and the data in Table 2 above the equation for the random variable theta is...

$$\theta = \left[\left(0.0953 - 0.0392 - \frac{1}{2} \times 0.2000^2\right) \times 5.00 + 0.2000 \times \sqrt{5.00} \times 0.75\right] \Big/ 5.00 = 0.1032 \tag{14}$$

Question 1: What is asset value at the end of year 5?

Using Equations (5) and (14) above and the data in Table 2 above the answer to the question is...

$$A_5 = 1,000,000 \times \text{Exp}\left\{0.1032 \times 5\right\} = 1,675,100 \tag{15}$$

Question 2: What are cumulative dividends and distributions over the time interval $[0, 5]$?

Using Equation (9) and (14) above and the data in Table 2 above the answer to the question is...

$$D_5 = \frac{0.0392}{0.1032} \times 1,000,000 \times \left[\text{Exp}\left\{0.1032 \times 5\right\} - 1\right] = 256,600 \tag{16}$$

Question 3: What is the dividends and distributions cash account balance at the end of year 5?

Using Equations (13) and (14) above and the data in Table 2 above the answer to the question is...

$$C_5 = \frac{0.0392}{0.1032 - 0.0296} \times 1,000,000 \times \text{Exp}\left\{0.0296 \times 5\right\} \times \left(\text{Exp}\left\{(0.1032 - 0.0296) \times 5\right\} - 1\right) = 274,800 \tag{17}$$